Math 53: Multivariable Calculus **Sections 201, 204**

Worksheet answers for 2021-11-17

If you would like clarification on any problems, feel free to ask me in person. (Do let me know if you catch any mistakes!)

Answers to computations

Problem 1. There is zero flux through the lateral side of ∂E because the vector field is vertical and thus perpendicular to $\mathbf{n}$ for that face. For the top face, $\mathbf{F} \cdot \mathbf{n} = 3$ at all points, so the flux is 12π. For the bottom face, $\mathbf{F} \cdot \mathbf{n} = -1$ at all points, so the flux is -4π. Altogether the net outwards flux is $\boxed{8\pi}$.

Problem 2. One possible choice of S is $z = x^2 - y^2$, $x^2 + y^2 \leq 1$. Note that C is traversed counterclockwise when viewed from above, and thus we need to give S the upwards orientation to have $\partial S = C$. With the standard parametrization $\mathbf{r}(x, y) = \langle x, y, x^2 - y^2 \rangle$ for S,

$$\mathbf{r}_x \times \mathbf{r}_y = \langle -2x, 2y, 1 \rangle$$

which is indeed upwards since the z-component is positive. So, letting D denote the region $x^2 + y^2 \leq 1$ in the xy-plane, our integral is

$$\begin{aligned}
\int_C \langle x^2 y, x^3/3, xz \rangle \cdot \mathrm{d}\mathbf{r} &= \iint_S (\nabla \times \langle x^2 y, x^3/3, xz \rangle) \cdot \mathrm{d}\mathbf{S} \\
&= \iint_S \langle 0, -z, 0 \rangle \cdot \mathrm{d}\mathbf{S} \\
&= \iint_D \langle 0, y^2 - x^2, 0 \rangle \cdot \langle -2x, 2y, 1 \rangle \, \mathrm{d}x \, \mathrm{d}y \\
&= \iint_D (2y^3 - 2x^2 y) \, \mathrm{d}x \, \mathrm{d}y \\
&= \int_0^{2\pi} \int_0^1 (2r^3 (\sin\theta)^3 - 2r^3 (\cos\theta)^2 \sin\theta) r \, \mathrm{d}r \, \mathrm{d}\theta \\
&= \int_0^{2\pi} \int_0^1 (2r^3 (\sin\theta)(1 - (\cos\theta)^2) - 2r^3 (\cos\theta)^2 \sin\theta) r \, \mathrm{d}r \, \mathrm{d}\theta.
\end{aligned}$$

The $\mathrm{d}r$ integral is easy, and the $\mathrm{d}\theta$ can be done by e.g. $u = \cos\theta$, $\mathrm{d}u = -\sin\theta \, \mathrm{d}\theta$.

Problem 3.

(a) Let's set up the integral:

$$\iint_S \frac{-\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{3/2}} \cdot \mathrm{d}\mathbf{S} = \iint_S -\langle x, y, z \rangle \cdot \mathrm{d}\mathbf{S}$$

where we have simplified the integral observing that $x^2 + y^2 + z^2 = 1$ on S. If we use the fact that $\iint_S 1 \, \mathrm{d}S = 4\pi$, then we can proceed in the same way as Problem 2 from the November 12 worksheet. The final answer ends up just being $\boxed{-4\pi}$.

(b) If it were possible to write $\nabla \times \mathbf{G} = \mathbf{F}$ for some vector field $\mathbf{G}$, then we could apply Stokes' Theorem to obtain

$$\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iint_S (\nabla \times \mathbf{G}) \cdot \mathrm{d}\mathbf{S} = \int_{\partial S} \mathbf{G} \cdot \mathrm{d}\mathbf{r} = 0$$

because $\partial S = \varnothing$ (the sphere has no boundary!). But this contradicts the answer from (a).

Problem 4. Let S denote the part of the surface $z = xy$ that is enclosed by C, with the downwards orientation (so that $\partial S = C$ as oriented curves). Then Stokes' Theorem gives

$$\begin{aligned}
\int_C \langle yz, xz + yz - 2z, xy + x^2/2 \rangle \cdot \mathrm{d}\mathbf{r} &= \iint_D \langle 2 - y, -x, 0 \rangle \cdot \mathrm{d}S \\
&= \iint_D \langle 2 - y, -x, 0 \rangle \cdot \langle y, x, -1 \rangle \, \mathrm{d}x \, \mathrm{d}y \\
(*) \qquad &= \iint_D (2y - y^2 - x^2) \, \mathrm{d}x \, \mathrm{d}y
\end{aligned}$$

where D is the projection of S into the xy-plane. Note that the $\langle y, x, -1\rangle$ comes from letting $\mathbf{r}(x, y) = \langle x, y, xy\rangle$ and computing $\mathbf{r}_y \times \mathbf{r}_x$ for a downwards normal.

This last integral is maximized if D **is exactly the region on which the integrand is positive**, which ends up being a disk of radius 1 centered at $(0, 1)$:

$$2y - y^2 - x^2 \geq 0$$
$$1 \geq x^2 + (y-1)^2.$$

The corresponding curve C is then the curve cut out by the equations $x^2 + (y-1)^2 = 1$ and $z = xy$. Parametrically we can take

$$\boxed{\begin{aligned} x &= \sin t \\ y &= 1 + \cos t \\ z &= (\sin t)(1 + \cos t) \end{aligned}}$$

where $0 \leq t \leq 2\pi$. To compute the integral $(*)$ we can either switch to polar, i.e. $x = r\cos\theta$, $y = r\sin\theta$, or use a “shifted” polar parametrization as $x = r\cos\theta$, $y = 1 + r\sin\theta$. With the first approach, the integral becomes

$$\int_0^{\pi} \int_0^{2\sin\theta} (2r\sin\theta - (r\sin\theta)^2 - (r\cos\theta)^2) r \, \mathrm{d}r \, \mathrm{d}\theta.$$

With the second approach, the integral is nicer:

$$\int_0^{2\pi} \int_0^{1} (1 - (r\cos\theta)^2 - (r\sin\theta)^2) r \, \mathrm{d}r \, \mathrm{d}\theta = \boxed{\pi/2}.$$

(Note that the absolute value of the Jacobian determinant still ends up being r in the second parametrization—check this!).